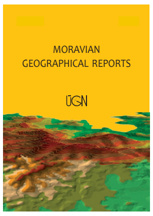

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Functional regions in gravity models and accessibility measures

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Abstract

Accessibility measures are useful for studies in Economic Geography. For example, accessibility to potential customers can be used in a study of firm behaviour. In such a study, it would be relevant to consider where potential customers live. This can be accomplished by splitting the accessibility measure into three parts: accessibility within the municipality, in other municipalities within the functional region, and in other regions. Many studies have proved this to be a very useful way to incorporate the spatial structure of the economy into economic studies. This paper deals with the issue of finding the distance-friction parameters needed to calculate such accessibility measures. There is a particular distance-friction parameter for interaction within the municipality, between municipalities within the functional region, and between regions. One way to find the distance-friction parameters is to solve a constrained gravity model, in which the functional regions are used as constraints. Both the models and the optimisation procedures in matrix form, and the Matlab programs used in the research are presented. The spatial constraints are gradually introduced into the models, which empowers the researcher to make such adjustments on their own. The data set used is available for downloading, and the reader can then try the models before creating a data set of their own.

Key words: *spatial interaction; commuting; gravity model; entropy; constrained optimisation; Matlab; Sweden*

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1. Introduction

On the global level, the use of specialisation and scale economies increase overall production. Individuals as well as regions specialise in producing only a part of what they consume. With more goods and services available, society has the potential to create a better life for the population. Transportation of both production factors and products are essential factors in this complex system.

It is important to consider and take into account that economic activity has a location, since spatial interaction in most cases declines with distance. Geurs and van Wee (2004) present and review accessibility measures: Hansen (1959) was one of the first to use the accessibility concept. Johansson, Klaesson and Olsson (2002, 2003) suggest that it is useful to split the accessibility measure into parts, and the idea of accessibility measures on three different spatial levels has been widely adopted. For example, it matters to a firm, with a store in a municipality, if a potential customer lives within the municipality, in another municipality within the functional region, or in another region. The firm can calculate accessibility to potential customers within the municipality, in other municipalities within the functional region, and in other regions. It can be valuable to split the accessibility in this way, since they are likely to be of unequal importance to the firm.

Many studies, mostly Swedish, have used the results from our earlier studies (Johansson, Klaesson and Olsson, 2002, 2003). It has been used to study many different activities: for example, Andersson and Ejermo (2005) study knowledge sources and the innovativeness of corporations; Gråsjö (2006) studies spatial spillovers of knowledge production; Karlsson and Olsson (2006) study how to define functional regions; Johansson and Karlsson (2007) study R&D and export diversity; Andersson and Gråsjö (2009) study representations of space in empirical models; Olsson (2012) studies the work at the public employment offices; Backman (2013) studies human capital and firm productivity; Larsson and Öner (2014) study retail location; and Larsson (2014) studies the density-wage relationship. Gråsjö and Karlsson (2015) is a nice review that contains additional papers. Gråsjö and Karlsson (2013: 1) write "*However, it is a general method and there is no reason why the method does not apply for other countries*".

In order to calculate accessibility at three different spatial levels, the corresponding distance-friction parameters are needed. The main purpose of this paper is to enable you to

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calculate the distance-friction parameters for the country you are interested in. The procedures are illustrated, and you will learn how to solve such models in detail using Matlab. In this paper, three models are stated in matrix form. This makes it easier to connect the text to the computer program. The ambition is to make it easy to look at the mathematical formulation and find almost the same in the program. In order to reduce the threshold, a data set is available for downloading. With the data and programs available, you can run the programs and check all the results. In this paper, the first model is gradually improved by incorporation of additional spatial constraints. There are several advantages with this approach. It makes the presentation cleaner and easier to grasp. Moreover, it enables you to make your own changes in the programs. In the future, you may want to estimate another version of the third model, or you may have a data set structured differently. After reading this paper you can handle such issues with ease. At least, that is the intention. The models are gradually made more complex, by adding constraints, to better reflect reality. It is also a purpose of the paper to present a comparison of the predictive power of the models. The third model has relatively many constraints, and performs better.

2. Commuting

Most workers have a relatively short commute, and it is rare to find a worker with a really long commute. This tendency is illustrated in Fig. 1. In this paper, municipalities are used as the spatial unit of analysis. The municipalities are more or less related to each other, however, and this relatedness across municipalities is captured using functional regions. It is possible to form functional regions using several approaches. The basic idea is that a functional region is built from municipalities with a relatively high level of interaction. In this paper the local labour market definition of a functional region is used. A local labour market consists of the municipalities that are tightly connected by commuting. A local labour market has a self-sufficient centre and surrounding municipalities. The surrounding municipalities are added to the core municipality, or to a municipality connected to the core, using one-way commuting. You find details of the procedure and maps of the Swedish local labour markets from Statistics Sweden (2015). An alternative to local labour markets is to create commuting zones using two-way commuting. Obviously, it is also possible to make other considerations. Karlsson and Olsson (2006) present local labour markets and some other methods and alternatives. The exact version of the functional region is not that important. The results will be similar if another version is picked. The basic reason is that most municipalities would be aggregated to the same functional region, independent of approach.

The commuting pattern gradually changes with time, and the area under the curve in Fig. 1 gradually shifts to the right with increased mobility. Not much happens to the pattern during a short period of time, but the pattern may change significantly if you observe a longer period. In Sweden, the daily average mobility of persons has increased from half a kilometer in the year 1900 to 45 kilometers in the year 1999 (Andersson and Strömquist, 1988; SIKA, 2000). The Swedish Institute for Transport and Communications Analysis (SIKA) has been replaced by the government agency Transport Analysis, and they estimate that the 2011 mobility is 44 kilometers (Transport Analysis, 2013). This change is also readily seen in the

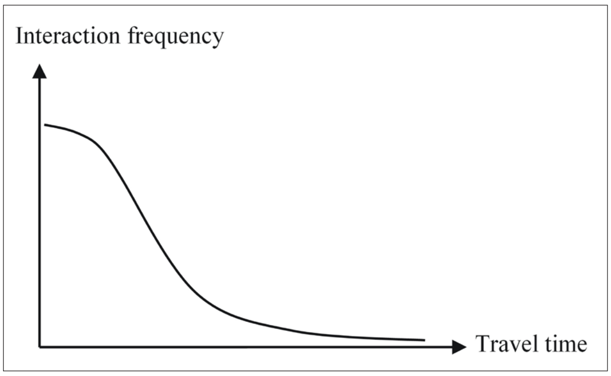

Fig. 1: Interaction declines with distance
Source: author´s elaboration

number of functional regions. With a long-term perspective, the number of functional regions has declined. This means that the models capture the spatial structure at a point in time. The overall tendencies will be the same for example ten years later, but the models should be rerun once in a while with the then present spatial structure. Moreover, it is possible to form functional regions per category of workers. In some studies, one may want to investigate educational, occupational, and/or gender differences.

The country consists of n municipalities, and a worker commutes from the home municipality, $i = 1, 2, \ldots, n$, to the work-place municipality, $j = 1, 2, \ldots, n$. The observed commuting information is collected in the $(n \times n)$ commuting matrix, $\mathbf{c} = \{c_{ij}\}$. A solution to a model would give the estimated commuting matrix, $\tilde{\mathbf{c}}$. There is also a corresponding $(n \times n)$ commuting-time matrix, $\mathbf{t} = \{t_{ij}\}$. Let us define a $(1 \times n)$ unit row vector, $\mathbf{u}$. The $(n \times 1)$ vector with the number of workers that lives in the municipalities equals the row sum of the commuting matrix, $\mathbf{o} = \mathbf{cu}'$, and the $(1 \times n)$ vector with the number of jobs in the municipalities equals the column sum of the commuting matrix, $\mathbf{d} = \mathbf{uc}$. The existing spatial structure is captured in the form of matrices. Three dummy variables are used to classify that a commute may end within the home municipality, in another municipality within the same functional region, or in another region. When a commute ends within the home municipality $k_{ij} = 1$, otherwise $k_{ij} = 0$. If a commute ends in another municipality within the home region $l_{ij} = 1$, otherwise $l_{ij} = 0$. If the commute ends in another region $m_{ij} = 1$, otherwise $m_{ij} = 0$. This information is collected in the $(n \times n)$ regional dummy matrices $\mathbf{k}$, $\mathbf{l}$, and $\mathbf{m}$, respectively. In this study, only links with a commuting time less than 150 minutes are included. This means that commuting on the other links, $(\mathbf{c}(\mathbf{t} > 150) = 0)$, are ignored. To identify all links that are included in this study zones are created, and collected into the $(n \times n)$ zone matrix, $\mathbf{z}$. In this matrix $z_{ij} = 1$ if $t_{ij} \leq 150$, otherwise $z_{ij} = 0$.

3. Data

In Table 1 you find the first five and last five rows in the Excel file used as input. The data set is in an Excel file that you have to download to run the Matlab programs. It is available from the following address: www.his.se/commuting. Nevertheless, it is useful to illustrate the structure of the data in this paper. At that time (1998), Sweden was separated in 289 municipalities. Hence, there are 83,521 commuting links. Each link has its own row in the Excel file. For each link, the data contains information whether the commute is within a municipality, between municipalities within a

Row	From	To	k_{ij}	l_{ij}	m_{ij}	t_{ij}	c_{ij}
1	1	1	1	0	0	4.71	6,904
2	1	2	0	1	0	11.52	141
3	1	3	0	1	0	24.31	28
4	1	4	0	1	0	30.98	11
5	1	5	0	1	0	16.10	479
…	…	…	…	…	…	…	…
83,517	289	285	0	0	1	209.64	225
83,518	289	286	0	0	1	236.85	3
83,519	289	287	0	0	1	216.61	19
83,520	289	288	0	0	1	219.08	2
83,521	289	289	1	0	0	67.13	10,549

Tab. 1: An excerpt from the data file, but the file only contains the white part
Source: Statistics Sweden and the Swedish Road Administration; author´s calculation

region, or between regions, the commuting time, and the number of commuters. The Excel file only contains the white part of Table 1.

The data has several sources. The commuting information originates from the Labor Statistics based on administrative sources (RAMS) from Statistics Sweden. Also, the information regarding the spatial structure originates from Statistics Sweden. The commuting times come from The Swedish Road Administration.

In Tab. 2, descriptive statistics for the number of commuters per link are presented. In Tab. 3, you find descriptive statistics for the commuting time per link. These numbers are calculated using only the active links with a commute shorter than 150 minutes. In Tab. 2, you also find the number of active links, the number of links with zero commuters, and the total number of links, given that the commuting time is shorter than 150 minutes. The number of commuters and the commuting times are clearly different for commutes within a municipality, between municipalities within a region, and between regions. In this paper, commuting flows are separated into commuting within a municipality, between municipalities within a region, and between regions. This separation is based on that these commuting flows differ. The null hypotheses, that the relative commuting frequencies, c_{ij} / o_i, for commuting within a municipality, between municipalities in a region, and commuting between regions are equal, have been tested and they are rejected.

4. Models, Matlab programs and results

Some spatial-interaction models are linear when written in logarithmic form. Fischer and Wang (2011) present the drawbacks related to the use of ordinary least squares to

Measure	Within municipality	Within region	Between regions	Sum
Min	761	1	1	–
Median	4,394	51	3	–
Mean	9,732	392	22	–
Max	266,980	19,647	6,050	–
Std. dev.	20,595	1,340	105	–
# active links*	289	2,087	9,911	12,287
# zero links*	0	81	8,413	8,494
# links*	289	2,168	18,324	20,781

Tab. 2: Descriptive statistics for commuting per link per commuting type
*Note: * Only commuting with $t < 150$ included*
Source: Statistics Sweden and the Swedish Road Administration; author´s calculation

Measure	Within municipality	Within region	Between regions
Min	3.4	6.3	12.3
Median	13.0	31.4	89.6
Mean	17.3	33.8	90.4
Max	89.8	96.3	150.0
Std. dev.	13.6	15.7	33.9

Tab. 3: Descriptive statistics for commuting time (minutes) per link per commuting type
Source: Statistics Sweden and the Swedish Road Administration; author´s calculation

estimate such a model. So, even though it may be tempting to estimate the logarithmic form of a spatial-interaction model using ordinary least squares, it should be avoided. In models of commuting it is preferred that the observed number of a) jobs in a municipality, and b) workers that live in a municipality (i.e. the data) both to be exactly equal to the estimates produced from the model. Olsson (2002) writes that constrained models have the advantage in that "(by construction) the model outcome is consistent with actual in- and out-commuting." In addition, we often want to include other constraints (e.g. time constraints). You can model the individual's choice to commute, or the aggregate commuting pattern. A gravity model of the aggregated commuting pattern relates interaction to an origin weight function, a destination weight function, and a distance deterrence function (Sen and Smith, 2011). The aggregate commuting function derived from maximising entropy is equivalent in form to the one derived from a logit model of individual (discrete) choice (Anas, 1983; Mattsson, 1984). So, studying the commuting pattern by maximising entropy a) produces a solution similar in form to the one that follows from individuals choosing their commute, and also b) enforces structure to the model via constraints.

In this paper, the aggregate commuting pattern is modelled by maximising entropy. In this section, you find three models of commuting. The first model has only two constraints. The point of this model is not that it will replicate the commuting pattern well, but that this model contains the essence of the following models. Each model is fully presented, i.e. the program used to estimate the model is described and the results are presented, before the next model is introduced. The first model is the base to which spatial structure (e.g. functional region, origin and destination constraints) is gradually incorporated. This is straight forward given an understanding of the first model and the Matlab program used to solve it. In the following models, the ideas presented in the first model are just extended. The second model has six constraints, and the third model has 582 unique constraints. The ambition is to incorporate spatial constraints into the model and to better replicate the pattern illustrated in Figure 1. If you want to get a preview of what is ahead, you can compare Fig. 1 to Fig. 10. It is the third model that is the best, since it enforces many more constraints. The first two models are just used to get to the third model, in the easiest possible manner.

By construction, Model 3 does a better job replicating the commuting pattern. In Model 3, the balancing factors (i.e. constraint multipliers) for where workers live (i.e. the origin constraints) and work (i.e. the destination constraints) captures the spatial surrounding of locations. All workers in Sweden are included in the data, but if all workers and all firms could redo their choices, many choices would change. The observed commuting data is but one realisation out of many possible. It is an aggregate observation in time of the (random) discrete choices made by individuals and firms. One consequence of this randomness is that there are spatial dependencies, e.g. if relatively many from a municipality commute on one link, it follows that relatively fewer commute on the other links. This would be seen as deviations from the estimated pattern.

4.1 Model 1

The observed population equals the sum of all commuters, $p = \sum_i \sum_j c_{ij} = \mathbf{ucu}'$. The Hadamard product sign, $\circ$, is used for entrywise multiplication of matrices. The observed total commuting time equals $r = \sum_i \sum_j c_{ij} t_{ij} = \mathbf{u(c \circ t)u}'$. In the first model, two constraints enforce that the estimated population, $\tilde{p} = \mathbf{u\tilde{c}u}'$, equals the observed population, and that the estimated total commuting time, $\tilde{r} = \mathbf{u}(\tilde{\mathbf{c}} \circ \mathbf{t})\mathbf{u}'$, equals the observed total commuting time. In constrained gravity models the objective is to maximise the system entropy, $\sum_i \sum_j c_{ij} \ln(c_{ij}) - c_{ij} = -\mathbf{u}(\tilde{\mathbf{c}} \circ \ln(\tilde{\mathbf{c}}) - \tilde{\mathbf{c}})\mathbf{u}'$, subject to the constraints. Therefore, the primal formulation of the problem is to max $\mathrm{L}(\tilde{\mathbf{c}}, \delta, \gamma)$, where the Lagrangian function is $\mathrm{L}(\tilde{\mathbf{c}}, \delta, \gamma) = -\mathbf{u}(\tilde{\mathbf{c}} \circ \ln(\tilde{\mathbf{c}}) - \tilde{\mathbf{c}})\mathbf{u}' + \delta(\mathbf{u\tilde{c}u}' - p) + \gamma(r - \mathbf{u}(\tilde{\mathbf{c}} \circ \mathbf{t})\mathbf{u}')$.

Let us call the Lagrangian multipliers, δ and γ, the proximity-preference parameter and the distance-friction parameter, respectively. In this model, the proximity-preference parameter is a fixed factor for all commutes, and does not really reveal any preference for proximity. But, the name will make more sense in the following models. The Lagrangian written in this form highlights the constraints. But, to get to the dual formulation of the problem it is easier to use $\mathrm{L}(\tilde{\mathbf{c}}, \delta, \gamma) = \mathbf{u}(\tilde{\mathbf{c}} \circ \ln(\tilde{\mathbf{c}}) + \tilde{\mathbf{c}} + \delta\tilde{\mathbf{c}} - \gamma\tilde{\mathbf{c}} \circ \mathbf{t})\mathbf{u}' - \delta p + \gamma r$. We can rewrite $\partial \mathrm{L}/\partial\tilde{\mathbf{c}} = -\ln(\tilde{\mathbf{c}}) + \delta - \gamma \mathrm{t} = 0$ as $\tilde{\mathbf{c}} = \exp(\delta\mathbf{u'u} - \gamma\mathbf{t})$. Hence, commuting on a particular link equals $c_{ij} = \exp(\delta - \gamma t_{ij})$. Inserting this in the primal form gives the dual form: $\min \mathrm{D}(\delta, \gamma)$, where $\mathrm{D}(\delta, \gamma) = \mathbf{u} \exp(\delta\mathbf{u}'\,\mathbf{u} - \gamma\mathbf{t})\mathbf{u}' - \delta p + \gamma r$.

The Newton-Raphson iterative procedure is used to find the optimum, and you find a description of the procedure in Appendix 1 (see link to Supplementary material at the end of the article). The iterative procedure needs some parameter start values. Reasonable start values must fulfill one of the constraints, and here the population constraint is used, $\mathbf{u\tilde{c}u}' - p = 0$. If $\gamma_0 = 0$ it follows that $\delta_0 = \ln(p / (\mathbf{uzu}'))$. In this study all links where $t_{ij} > 150$ are ignored. This reduces the number of links from $\mathbf{u(k + l + m)u}'$ which is 83,521 to $\mathbf{uzu}'$ which is 20,781. With p = 3,847,782 the start value is equal to $\delta_0 = 5.2212$. Now, it is time to iterate from the start values towards the solution. The start values imply that commuting is not affected by commuting time. Hence, estimated commuting on links with long commuting time is bigger than observed commuting. Therefore, the estimated commuting flows use more time than is allowed. This implies that the distance friction parameter has to be raised. Raising the distance friction reduces estimated commuting flows, which leads to that too few persons work. It gives that the proximity-preference parameter has to be raised. And, this is sequentially repeated until the solution is found. If a constraint is violated in the opposite direction, the parameter estimate is adjusted accordingly.

As said, it is most likely that the estimated commuting flows do not fulfill the constraint on commuting time, $r - \mathbf{u}(\tilde{\mathbf{c}} \circ \mathbf{t})\mathbf{u}' = 0$, at the start. The distance-friction parameter estimates are adjusted using the Newton-Raphson procedure. The partial derivatives are $\partial\mathrm{D} / \partial\gamma = -\mathbf{u}(\tilde{\mathbf{c}} \circ \mathbf{t})\mathbf{u}' + r = r - \tilde{r}$ and $\partial^2\mathrm{D} / \partial\gamma^2 = \mathbf{u}(\tilde{\mathbf{c}} \circ \mathbf{t} \circ \mathbf{t})\mathbf{u}' = \tilde{s}$, which leads to the following adjustment scheme $\gamma_{(\mathrm{n}+1)} = \gamma_\mathrm{n} - \rho(\mathrm{r} - \tilde{r}_\mathrm{n}) / \tilde{s}_\mathrm{n}$. It is important to recalculate the commuting flows, before adjusting the proximity-preference parameter. The derivatives are $\partial\mathrm{D} / \partial\delta = \mathbf{u\tilde{c}u}' - p = \tilde{p} - p$ and $\partial^2\mathrm{D} / \partial\delta^2 = \mathbf{u\tilde{c}u}' = \tilde{p}$, which leads to the following adjustment scheme, $\delta_{(\mathrm{n}+1)} = \delta_\mathrm{n} - \rho(\tilde{p}_n - p) / \tilde{p}_n$. In the first model, $\rho = 1$. It is important to recalculate the commuting flows, before starting over again. The program iterates until all constraints are fulfilled with extreme accuracy, since the run time is short.

4.1.1 The Matlab program

Now it is time to look at the Matlab program for Model 1. To make the reading easier, the program is included in

Appendix 2 (see Supplementary material). The structure of the first program is maintained in the models to come. First, the data file is read. In this section of the program **u**, **t**, and **c** are declared, and filled with values from the data. Then, the a priori information is calculated from the data, and the start values are set. In this part of the program, r, p, $\mathbf{z}$, and the start values are calculated as described in the text above. Here, the estimated commuting flows using the parameter start values are calculated. The parameter start values and the value of the dual function are saved. This is done to later illustrate convergence. In the main iterative part of the program, each parameter is adjusted in relation to the constraint deviation. First, the distance-friction parameter is adjusted. Second, the proximity-preference parameter is adjusted. After each parameter adjustment the estimated commuting flows are recalculated. The new parameter values and the value of the dual function are saved. The end part of the program creates graphs, and saves the results to an Excel file. The Model 1 program is adjusted in the following models to incorporate additional spatial information.

The mathematical notation in the program is for the most part as in the text, so it should be easy to follow. However, there are four minor exceptions. In the text the Hadamard product sign $\circ$ is used for entrywise multiplication of matrices. In Matlab .* multiply two matrices entrywise. The other three types of exceptions are illustrated by example. The proximity-preference parameter is delta in the program and δ in the text. The travel-time matrix is **t** in the text and t in the program. In the text $\tilde{p}$ refers to the estimated working population, while p_tilde is used in the program.

The program is published on the following web address: *www.his.se/commuting*. This means that you do not have to retype the code to run the program, you can just use the published file. In order to run the program for the first model you must save the data and the program to your computer. It is recommended that you first save the Excel file to your Matlab folder. In the next step, you save the program file containing the first program into the same Matlab folder. Then start Matlab and run the program.

4.1.2 Results

In Figure 2 you find the estimated distance-friction parameter per iteration. In Figure 3 you find the estimated proximity-preference parameter per iteration. To keep the first Matlab program as simple as possible the value of the dual function and the parameter values are collected per iteration in the published program. The start values of the distance-friction parameter and the proximity-preference parameter is zero and 5.2212, respectively. This gives the start point (5.2212,0) in Figure 4. In Figure 4, the thick line illustrates the path from the start point to the solution. The value of the dual function per iteration is presented in Figure 5. After about 15 iterations neither the parameters nor the value of the dual function change more than marginally. The model converges at the solution, where the distance-friction parameter is 0.1197 and the proximity-preference parameter is 9.809.

However, nothing prevents us from saving all information during the approach to the solution. By doing some small adjustments in the Matlab program, it is possible to save the parameter values and the value of the dual function at every parameter adjustment, rather than per iteration. From the start point (5.2212,0), the distance friction parameter is adjusted to 0.0075, leading to the point (5.2212,0.0075) in Figure 4. Then the estimated commuting flows are recalculated and the proximity-preference parameter is adjusted to 6.1519, leading to the point (6.1519,0.0075). This ends the first iteration, and is seen as the first step from the start point following the thin line in Figure 4. Hence, iterating and saving results in this way gives a set of steps to the solution. It is of course also an option to just save the final solution values.

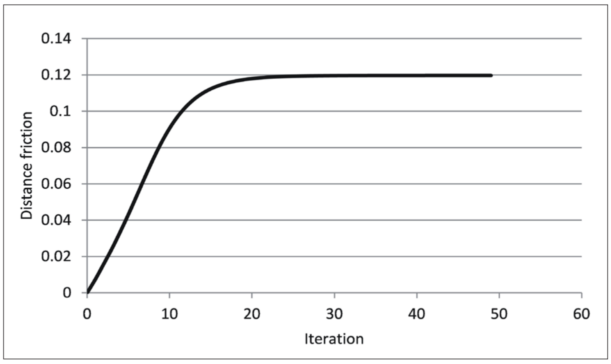

Fig. 2: Distance-friction parameter convergence
Source: author´s elaboration

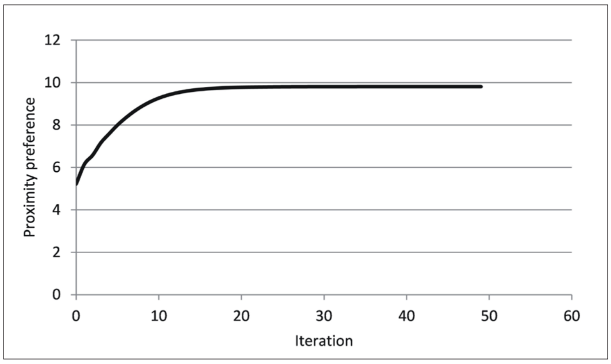

Fig. 3: Proximity-preference parameter convergence
Source: author´s elaboration

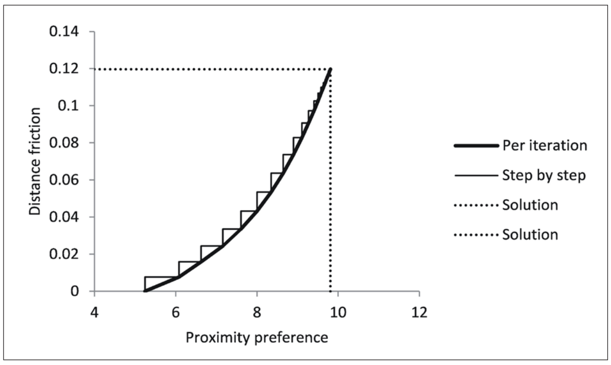

Fig. 4: The two ways to the solution
Source: author´s elaboration

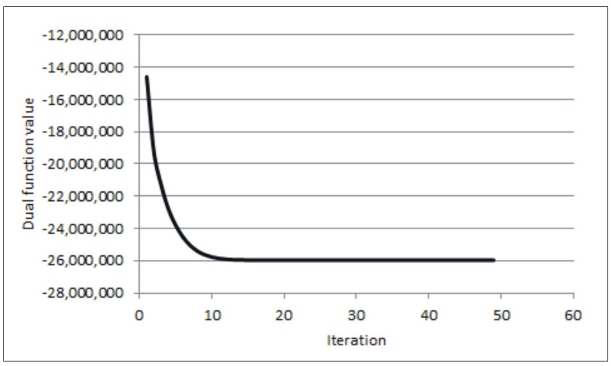

Fig. 5: The value of the dual function per iteration
Source: author´s elaboration

4.2 Model 2

Model 2 has six constraints, and they are similar to the two constraints present in Model 1. The idea is to incorporate more spatial information into the model. Some persons work within their home municipality, while others commute to another municipality within their home region, and some even commute to another region. Model 2 has three constraints replacing the Model 1 constraint enforcing that the estimated working population is equal to the observed population. The observed number of commuters within a municipality is equal to $p_1 = \mathbf{u}(\mathbf{k} \circ \mathbf{c})\mathbf{u}'$. The observed number of commuters between municipalities within the home region is $p_2 = \mathbf{u}(\mathbf{l} \circ \mathbf{c})\mathbf{u}'$. The observed number of commuters between regions is $p_3 = \mathbf{u}(\mathbf{m} \circ \mathbf{c})\mathbf{u}'$. They are collected in the column vector $\mathbf{p}$. In this study the working population is divided such that p_1 = 2,812,614, p_2 = 817,802 and p_3 = 217,366. To each constraint there is a proximity-preference parameter, all collected in the column vector δ. In Model 2, three constraints replace the constraint regarding total commuting time present in Model 1. The observed total commuting time for commutes within a municipality equals $r_1 = \mathbf{u}(\mathbf{k} \circ \mathbf{c} \circ \mathbf{t})\mathbf{u}'$. The observed total commuting time for commutes between municipalities within the home region is $r_2 = \mathbf{u}(\mathbf{l} \circ \mathbf{c} \circ \mathbf{t})\mathbf{u}'$. The observed total commuting time for commutes between regions is $r_3 = \mathbf{u}(\mathbf{m} \circ \mathbf{c} \circ \mathbf{t})\mathbf{u}'$. They are collected in the column vector $\mathbf{r}$. To each time constraint there is a distance-friction parameter, and they are collected in the column vector γ. This model is like splitting Model 1 into three completely separate parts. The primal form of the problem is max $\mathrm{L}(\tilde{\mathbf{c}}, \delta, \gamma)$, where $\mathrm{L}(\tilde{\mathbf{c}}, \delta, \gamma) = \sum_{s=0}^{6} \mathrm{L}_s$ and the Lagrangian parts L_s are defined in (1)–(7).

$$\mathrm{L}_0 = -\mathbf{u}(\tilde{\mathbf{c}} \circ \ln(\tilde{\mathbf{c}}) - \tilde{\mathbf{c}})\mathbf{u}' \quad (1)$$

$$\mathrm{L}_1 = \delta_1(\mathbf{u}(\mathbf{k} \circ \tilde{\mathbf{c}})\mathbf{u}' - p_1) \quad (2)$$

$$\mathrm{L}_2 = \delta_2(\mathbf{u}(\mathbf{l} \circ \tilde{\mathbf{c}})\mathbf{u}' - p_2) \quad (3)$$

$$\mathrm{L}_3 = \delta_3(\mathbf{u}(\mathbf{m} \circ \tilde{\mathbf{c}})\mathbf{u}' - p_3) \quad (4)$$

$$\mathrm{L}_4 = \gamma_1(r_1 - \mathbf{u}(\mathbf{k} \circ \tilde{\mathbf{c}} \circ \mathbf{t})\mathbf{u}') \quad (5)$$

$$\mathrm{L}_5 = \gamma_2(r_2 - \mathbf{u}(\mathbf{l} \circ \tilde{\mathbf{c}} \circ \mathbf{t})\mathbf{u}') \quad (6)$$

$$\mathrm{L}_6 = \gamma_3(r_3 - \mathbf{u}(\mathbf{m} \circ \tilde{\mathbf{c}} \circ \mathbf{t})\mathbf{u}') \quad (7)$$

You find the three constraints for the number of commuters in (2)–(4) and the three constraints on total commuting time in (5)–(7). This is similar to the earlier model, and the adjustment process to find the six Lagrangian multipliers is therefore straight forward. The derivative of the Lagrangian with respect to commuting gives the estimated commuting matrix $\tilde{\mathbf{c}} = \exp(\delta_1\mathbf{k} + \delta_2\mathbf{l} + \delta_3\mathbf{m} - (\gamma_1\mathbf{k} + \gamma_2\mathbf{l} + \gamma_3\mathbf{m}) \circ \mathbf{t})$. By inserting this into the Lagrangian we get the dual formulation of the problem, min $\mathrm{D}(\delta,\gamma)$, where $\mathrm{D}(\delta,\gamma) = \mathbf{u} \exp(\delta_1\mathbf{k} + \delta_2\mathbf{l} + \delta_3\mathbf{m} - (\gamma_1\mathbf{k} + \gamma_2\mathbf{l} + \gamma_3\mathbf{m}) \circ \mathbf{t})\mathbf{u}' - \delta'\mathbf{p} + \gamma'\mathbf{r}$.

To find reasonable start values, assume that all distance-friction parameters are zero and choose to enforce the three constraints regarding the number of commuters within the home municipality, between municipalities within the home region, and between the regions. Then the start values for the proximity preferences are $\delta_1 = \ln(p_1 / \mathbf{u}(\mathbf{k} \circ \mathbf{z})\mathbf{u}'))$, $\delta_2 = \ln(p_2 / (\mathbf{u}(\mathbf{l} \circ \mathbf{z})\mathbf{u}'))$, and $\delta_3 = \ln(p_3 / (\mathbf{u}(\mathbf{m} \circ \mathbf{z})\mathbf{u}'))$, respectively. If you compare these start values to the start value in Model 1 you see the similarity. Collect the derivatives $\tilde{s}_1 = \mathbf{u}(\mathbf{k} \circ \mathbf{t} \circ \mathbf{t} \circ \tilde{\mathbf{c}})\mathbf{u}'$, $\tilde{s}_2 = \mathbf{u}(\mathbf{l} \circ \mathbf{t} \circ \mathbf{t} \circ \tilde{\mathbf{c}})\mathbf{u}'$, and $\tilde{s}_3 = \mathbf{u}(\mathbf{m} \circ \mathbf{t} \circ \mathbf{t} \circ \tilde{\mathbf{c}})\mathbf{u}'$ in the column vector $\tilde{\mathbf{s}}$. Then the friction vector is adjusted using $\gamma_{(n+1)} = \gamma_n - \rho(\tilde{\mathbf{r}}_n - \mathbf{r})./\,\tilde{\mathbf{s}}_n$, where ./ is the symbol for piecewise division. The estimated commuting flows are recalculated before adjusting the proximity-preferences using $\delta_{(n+1)} = \delta_n - \rho(\tilde{\mathbf{p}}_n - \mathbf{p})./\,\tilde{\mathbf{p}}_n$. Also in Model 2 $\rho = 1$. Before iterating, the estimated commuting flows are recalculated once more.

4.2.1 The Matlab program

You find the program for Model 2 in Appendix 3 (see Supplementary material), and it is also available for downloading at www.his.se/commuting. The overall structure of the program is the same as for Model 1. However, Model 2 uses more spatial information. Therefore the **k**, **l**, and **m** matrices are also read from the Excel file. With them the new necessary vectors **p** and **r** are calculated. In the main part of the program, the parameters are adjusted. First, the distance-friction vector is adjusted in relation to the relevant constraint deviation. In this part **s** is calculated. Second, the proximity-preference vector is adjusted. This is the same as the adjustment procedure used in Model 1. A comment on notation: In the text for example $\tilde{s}_2$ refers to the second value in $\tilde{\mathbf{s}}$. In the Matlab program s_tilde(2) does that job. This is the principle used for any vector or matrix.

4.2.2 Results

In Figure 6 you find the distance-friction parameters per iteration. In Figure 7 you find the proximity-preference parameters per iteration. At the start the distance-friction parameters are set to zero, and the proximity-preferences are 9.1832, 5.9328, and 2.4734, for commuting within a municipality (i.e. local), commuting between municipalities within a region (i.e. regional) and between regions, respectively. The solution for the distance-friction parameters are 0.0294, 0.1027, and 0.0483. The solutions for the proximity-preference parameters are 9.6335, 8.5289, and 6.1309.

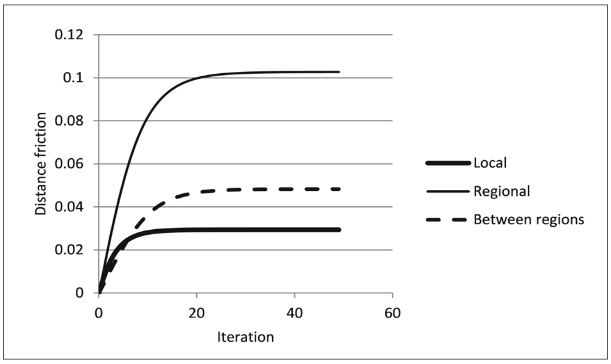

Fig. 6: Distance-friction parameter convergence
Source: author´s elaboration

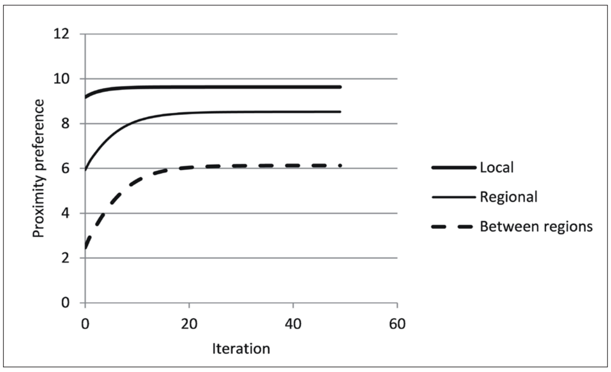

Fig. 7: Proximity-preference parameter convergence
Source: author´s elaboration

In Figure 8 you find the proximity preference and distance friction pairs from the start to the solution. Here the convergence process starts from points along the x-axis. In Fig. 9, you find the value of the dual function per iteration.

The solution is found after about 15 iterations. Then nothing much happens to the parameters and the value of the dual function. At the solution, the value of the dual function is smaller for Model 2 compared to the value for Model 1. This is expected, since Model 2 enforces more constraints. The first two models are only presented as the way to the final model. However, we can compare the results from Model 1 and Model 2 anyway. Both the proximity-preference parameter and the distance-friction parameter are higher in Model 1. Model 1 replicates the commuting pattern (Fig. 1) with one exponential function. The proximity-preference parameter is related to the intersection with the y-axis. The distance-friction parameter is related to the decline of commuting as commuting time is increased. This is illustrated in in Figure 10 by the dotted line. Model 2 replicates the pattern using three exponential functions. This is seen in Figure 10 as one solid line for commuting within the home municipality, one solid line for commuting between municipalities within the home region, and one solid line for commuting between regions. Model 3 has 582 unique constraints. In that model, the pattern is replicated using 20,781 (out of maximum 83,521) exponential functions.

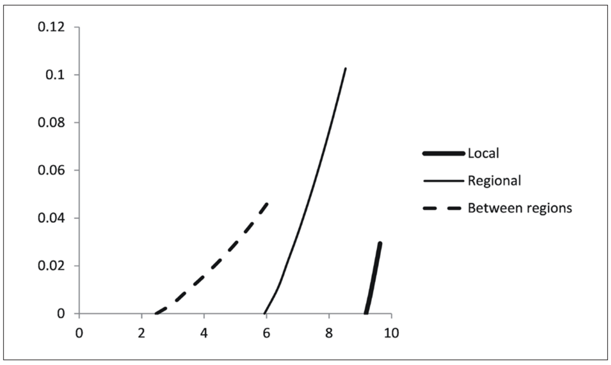

Fig. 8: The paths to the solution
Source: author´s elaboration

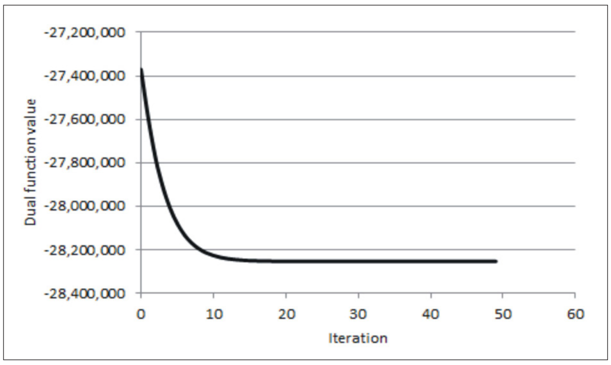

Fig. 9: The value of the dual function per iteration
Source: author´s elaboration

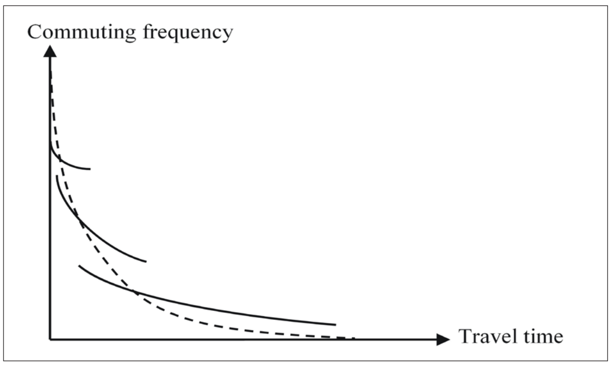

Fig. 10: The principal result of Model 1 and Model 2
Source: author´s elaboration

4.3 Model 3

Model 2 has three distance-friction parameters and three proximity-preference parameters. Those parameters (and constraints) are also present in Model 3, but in addition Model 3 also has commuting origin- and destination constraints. The third model is max $L(\tilde{\mathbf{c}}, \alpha, \beta, \delta, \gamma) = \sum_{s=0}^{8} L_s$, where the Lagrangian parts, L_s, are found in (8)–(16). Model 2 has three constraints for the amount of commuting, and they are included in the same way in Model 3, (11)–(13). Model 2 has three time constraints, and they are included in the same way in Model 3, (14)–(16). Model 3 in addition enforces that the estimated number of workers that live in each municipality is equal to the observed number, $\mathbf{o} = \mathbf{cu}' = \tilde{\mathbf{c}}\mathbf{u}'$. This adds 289 origin constraints, (9). However, only 288 origin constraints provide new information. The three constraints on the number of commuters together enforce that the estimated number of commuters is equal to the observed working population. This makes the 289th origin constraint redundant, since it will be enforced by the other constraints. To each origin constraint there is a Lagrangian multiplier which is called a push factor. They are collected in the column vector α. Because of programming convenience all 289 destination constraints are used, but one origin is used as base, here $\alpha_1 = 0$. Model 3 also enforces that the estimated number of jobs in each municipality is equal to the observed number of jobs, $\mathbf{d} = \mathbf{uc} = \mathbf{u}\tilde{\mathbf{c}}$. This adds 289 destination constraints, (10). As for the origin constraints, one of the destination constraints is redundant, since only 288 destination constraints provide information. To each destination constraint there is a Lagrangian multiplier which is called a pull factor. They are collected in the row vector β. Because of programming convenience all 289 destination constraints are used, but one pull factor is used as base, here $\beta_1 = 0$. The complete model now has 582 constraints. This is the setup in the Matlab program in Appendix 4 (see Supplementary material).

$$L_0 = -\,\mathbf{u}(\tilde{\mathbf{c}} \circ \ln(\tilde{\mathbf{c}}) - \tilde{\mathbf{c}})\mathbf{u}' \quad (8)$$

$$L_1 = \mathbf{u}(\alpha \circ (\tilde{\mathbf{c}}\mathbf{u}' - \mathbf{o})) \quad (9)$$

$$L_2 = (\beta \circ (\mathbf{u}\tilde{\mathbf{c}} - \mathbf{d}))\mathbf{u}' \quad (10)$$

$$L_3 = \delta_1(\mathbf{u}(\mathbf{k} \circ \tilde{\mathbf{c}})\mathbf{u}' - p_1) \quad (11)$$

$$L_4 = \delta_2(\mathbf{u}(\mathbf{l} \circ \tilde{\mathbf{c}})\mathbf{u}' - p_2) \quad (12)$$

$$L_5 = \delta_3(\mathbf{u}(\mathbf{m} \circ \tilde{\mathbf{c}})\mathbf{u}' - p_3) \quad (13)$$

$$L_6 = \gamma_1(r_1 - \mathbf{u}(\mathbf{k} \circ \tilde{\mathbf{c}} \circ \mathbf{t})\mathbf{u}') \quad (14)$$

$$L_7 = \gamma_2(r_2 - \mathbf{u}(\mathbf{l} \circ \tilde{\mathbf{c}} \circ \mathbf{t})\mathbf{u}') \quad (15)$$

$$L_8 = \gamma_3(r_3 - \mathbf{u}(\mathbf{m} \circ \tilde{\mathbf{c}} \circ \mathbf{t})\mathbf{u}') \quad (16)$$

The program needs reasonable start values. It is assumed that all distance-friction parameters, push- and pull factors are zero. The start values for the proximity preferences are $\delta_1 = \ln(p_1 / \mathbf{u}(\mathbf{k} \circ \mathbf{z})\mathbf{u}'))$, $\delta_2 = \ln(p_2 / (\mathbf{u}(\mathbf{l} \circ \mathbf{z})\mathbf{u}'))$, and $\delta_3 = \ln(p_3 / (\mathbf{u}(\mathbf{m} \circ \mathbf{z})\mathbf{u}'))$, respectively, which is exactly the same as is used in Model 2.

The partial derivative of the Lagrangian with respect to commuting gives the estimated commuting matrix $\tilde{\mathbf{c}} = \exp(\alpha\mathbf{u} + \mathbf{u}'\beta + \delta_1\mathbf{k} + \delta_2\mathbf{l} + \delta_3\mathbf{m} - (\gamma_1\mathbf{k} + \gamma_2\mathbf{l} + \gamma_3\mathbf{m}) \circ \mathbf{t})$ Inserting this into the Lagrangian gives the dual form, $\min D(\alpha,\beta,\gamma,\delta)$, where $D(\alpha,\beta,\gamma,\delta) = \mathbf{u} \exp(\alpha\mathbf{u} + \mathbf{u}'\beta + \delta_1\mathbf{k} + \delta_2\mathbf{l} + \delta_3\mathbf{m} - (\gamma_1\mathbf{k} + \gamma_2\mathbf{l} + \gamma_3\mathbf{m}) \circ \mathbf{t})\ \mathbf{u}' - \alpha'\mathbf{o} - \beta\mathbf{d}' - \delta'\mathbf{p} + \gamma'\mathbf{r}$.

Model 3 has four groups of parameters, and each group is adjusted separately. In the program the push factors are adjusted first. The origin constraints are constraints on the number of commuters. In that way they are similar to the three constraints on the number of commuters within the home municipality, between municipalities within the home region, and between regions. Therefore, how to adjust the push factors are easily inferred. The push factors are adjusted using $\alpha_{(n+1)} = \alpha_n - \rho(\tilde{\mathbf{o}}_n - \mathbf{o})./\,\tilde{\mathbf{o}}_n$. After recalculating the estimated commuting flows the pull factors are adjusted, $\beta_{(n+1)} = \beta_n - \rho(\tilde{\mathbf{d}}_n - \mathbf{d})./\,\tilde{\mathbf{d}}_n$, in a similar way. The estimated commuting flows are recalculated before the distance-friction vector is adjusted. At the end of the iteration, the proximity-preference vector is adjusted and the estimated commuting flows are recalculated once more. The distance-friction vector and the proximity-preference vector are adjusted as described above in Model 2. Compared to the previous models, Model 3 is more complex. In Model 3 the number of constraints is larger, and the constraints are interwoven. For that reason the relative adjustment factor is reduced for convergence, $\rho = 0.2$. It is possible to rerun the program for other adjustment factors, and trace the way to the solution in each case.

4.3.1 The Matlab program

You find the Matlab program for Model 3 in Appendix 4 (see Supplementary material). This program is also available to download from www.his.se/commuting. The program has grown to include the adjustment of the push and pull factors. In the main part of the program, the parameters are adjusted. First, the push factors are adjusted. Second, the pull factors are adjusted. Third, the distance-friction parameter vector is adjusted. Fourth, the proximity-preference parameter vector is adjusted. After a set of parameters has been adjusted, the estimated commuting flows are recalculated. In the program it is convenient to keep all 289 push factors and 289 pull factors. Hence, all 289 factors are adjusted using the same procedure, but then one of each factor is set to zero.

4.3.2 Results

At the solution, the distance-friction parameter for commuting within a municipality is 0.0248, the distance-friction parameter for commuting between municipalities within a region is 0.0958, and the distance-friction parameter for commuting between regions is 0.0514. You find the convergence process for the distance-friction parameters in Figure 11.

At the solution, the proximity-preference parameter for commuting within a municipality is 8.5147, the proximity-preference parameter for commuting between municipalities within a region is 7.4679, and the proximity-preference parameter or commuting between regions is 5.4938. The convergence processes for the proximity-preference parameters are illustrated in Figure 12. In Figure 13 the proximity-preference parameter and distance-friction parameter pairs from the start (along the x-axis) to the solution are illustrated. In the background, the 288 push and 288 pull factors are adjusted as well.

You find the value of the dual function per iteration in Fig. 14. Little happens to the parameter values and value of the dual function after 200 iterations. However, by iterating more the solution is pinpointed. The program is set to do 500 iterations. The solution value of the dual function

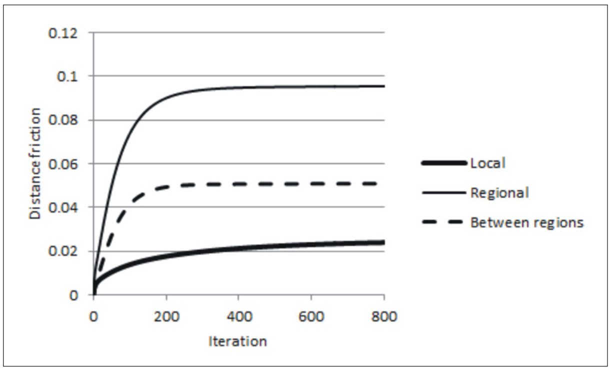

Fig. 11: Distance-friction parameter convergence
Source: author´s elaboration

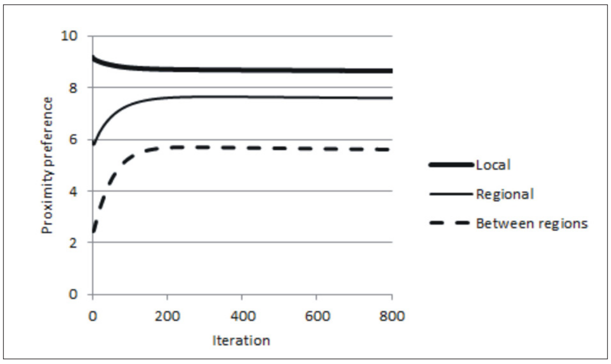

Fig. 12: Proximity-preference parameter convergence
Source: author´s elaboration

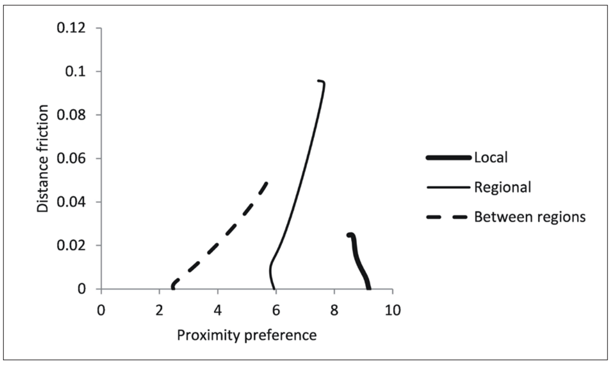

Fig. 13: The paths to the solution
Source: author´s elaboration

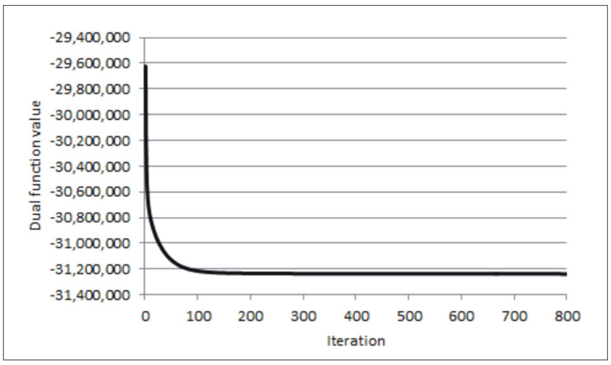

Fig. 14: The value of the dual function per iteration
Source: author´s elaboration

is smaller for Model 3. This is as expected, since Model 3 enforces many additional constraints.

4.4 Model 3 alternatives

In Model 3 the first municipality is used as base case, and hence both α_1 and β_1 are set to zero. Then, every other parameter is estimated in relation to them. Of course, any other municipality could serve as base. Using another municipality as base implies setting two other parameters to zero. This would give other solution push factors, pull factors, and proximity-preference parameters. It is their combined effect that is interesting. The combined effect remains the same in all cases. Moreover, the distance-friction vector is the same in all cases.

Another alternative would be to for example set β_1 and δ_3 to zero. Then the model has 289 origin constraints, and 289 push factors. To find start values all pull factors, all distance-friction parameters, and the two proximity-preference parameters are set to zero. Then it follows that $\alpha_0 = \ln(\mathbf{o}/289)$ in the case that all destinations are included for all origins. In this study the commuting time from a municipality should be smaller than 150 minutes in order for a destination to be included in the commuting zone. Then the start values are $\alpha_0 = \ln(\mathbf{o}./(\mathbf{zu}'))$. Obviously, you could have chosen another of the pull factors and push factors instead of β_1 and another of the proximity preferences instead of δ_3. This would give other push factors, pull factors, and proximity preference parameters. However, their combined effect is the same, nevertheless. The proximity-preference vector differs between model set ups, however the proximity-preference parameter differences are maintained in all set ups. Moreover, the distance-friction vector is the same in all cases.

Sometimes you see studies that set no parameter value to zero, i.e. all constraints are used, even though in Model 3 two constraints contain no new information. This works, since the parameters are estimated in relation to each other. In such a case, there are several sets of feasible start values to choose from. The solution push factors, pull factors, and proximity-preference parameters change with start values. However, this procedure gives the same parameter estimates for distance friction and proximity-preference parameter differences. Still, it is not good practice to include constraints with no information. Under such circumstances one needs to be careful when interpreting the results. When that is done properly, you find that the results are the same as you get if you only use constraints with real information.

4.5 Model comparisons

In this paper, three models of commuting have been presented. The idea was to start from a simple model, and gradually add spatial constraints to the model to better capture reality. It is interesting to see how well the models estimate the observed commuting pattern.

In Tables 4–6, you find descriptive statistics for (c.)□c □ for Model 1–3, respectively. For example, the median value for observed commuting as a share of estimated commuting within a municipality is 1.7, for Model 1 (Tab. 4). The corresponding median values for Model 2 and Model 3 are 0.5 (Tab. 5) and 1.0 (Tab. 6), respectively. Note that the median gets closer and closer to one. This is the case also for commuting between municipalities within a region, and for commuting between regions. It is also the case that the standard deviation is smaller in Model 3 than in Model 1. Model 3 has many more constraints and therefore performs better. This is also seen by that the means converge.

In Table 6, the standard deviation is relatively large for commuting between regions. One possible explanation for this is that there are some links that deviate from the pattern due to commuting by other means than car, i.e. train. Such flows are not accurately captured in this model.

5. Discussion and conclusion

Given the third model, you can create a version of the model by altering the set-up. You could for example just use one commuting time constraint instead of three. That means that you estimate only one distance-friction parameter. You could also remove the constraints for the number of commuters within a municipality, between municipalities within a region, and between regions. That means that you estimate no proximity-preference parameters. In such a version of the model, you must allow all 289 push- and pull-factors to adjust using the described procedure. The resulting distance-friction parameter is 0.1406. For this version of the model the median number of $\mathbf{c} / \tilde{\mathbf{c}}$ is 2.0, 0.5, 31.0 for commuting within a municipality, between municipalities within a region, and between regions, respectively. This can be compared to the corresponding numbers for Model 3 in Table 6.

As expected, Model 3 outperforms a version of the model using less constraints. It is also possible to alter Model 3 in other ways. Model 3 uses 20,781 links out of the maximum 83,521 links, and some of those links are not active (Table 2). It is straight forward to change the code such that only the 12,287 active links out of the 20,781 links

Measure	Within municipality	Within region	Between regions
Min	0.2	0.0	0.0
Median	1.7	0.1	8.9
Mean	62.0	0.7	1,298
Max	6,472	59.4	655,370
Std. dev.	447.6	2.7	11,985

Tab. 4: Descriptive statistics for the predictive performance of Model 1. Source: author´s calculation

Measure	Within municipality	Within region	Between regions
Min	0.1	0.0	0.0
Median	0.5	0.3	0.7
Mean	1.0	1.2	3.3
Max	21.6	90.5	687.8
Std. dev.	1.8	3.9	16.7

Tab. 5: Descriptive statistics for the predictive performance of Model 2. Source: author´s calculation

Measure	Within municipality	Within region	Between regions
Min	0.8	0.0	0.0
Median	1.0	0.6	0.8
Mean	1.1	0.9	2.6
Max	2.8	31.4	661.1
Std. dev.	0.2	1.5	10.5

Tab. 6: Descriptive statistics for the predictive performance of Model 3. Source: author´s calculation

are used ($\tilde{\mathbf{c}}(\mathbf{c} = = 0) = 0$). The resulting estimates for the distance-friction parameters do not differ much between these two versions of Model 3. Nothing prevents us from adding more constraints to Model 3. You could add, for example, housing expenditure and income constraints. Such models and results are discussed in Olsson (2015).

Hopefully, this paper has stimulated you into modelling spatial interaction. When a model like Model 3 has been solved, you have a set of distance-friction parameters. With them you can calculate accessibility measures to incorporate spatial aspects into different types of studies (e.g. the literature presented in the Introduction). Let us assume that we want to look at the accessibility to workers. This would be one important variable to consider when studying, for example, how easy it is to find someone to fill a vacancy. Johansson et al. (2002, 2003) suggest that one separates the total accessibility into three parts: accessibility within the municipality, accessibility in other municipalities within the region, and accessibility in other regions. Such spatial decomposition of the total accessibility is useful in empirical studies, since they likely are of unequal importance. In this study, the commuting pattern was in focus. But not all persons work, e.g. the unemployed, the retired, students, etc. The non-working part of the population also interacts spatially. Although not part of this study, such spatial interactions are also interesting to model. Moreover, those individuals are often included in the accessibility measures (e.g. as potential workers or customers, depending on the focus of the study). Such a spatial decomposition, moreover, does not acknowledge that competition also varies across locations. Geurs and van Wee (2004) identified several ways to introduce competition aspects into the accessibility measures, and one way would be to use the balancing factors of the solution to the gravity model (from α and β).

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